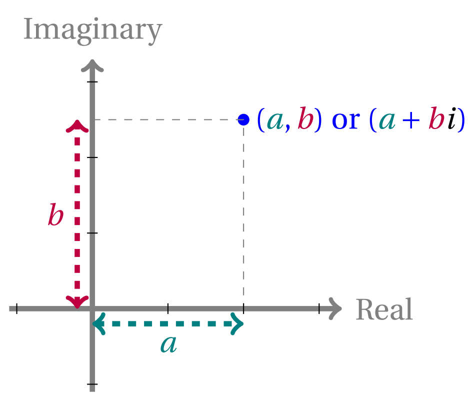
3.1: Complex Numbers

- Fun Fact: Complex numbers came to existence because mathematicians were interested in factoring all polynomial linearly. Now we use complex numbers in many areas of engineering.
- i is the square root of -1. ($i = \sqrt{-1}$)
- That is, i is a number such that $i^2 = -1$.
- An imaginary number is the principal square root of a negative number. If $-r < 0$, is negative, then the principal square root of $-r$ is $\sqrt{-r} = i\sqrt{r}$ is an imaginary number. Example $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$ or $\sqrt{-2} = \sqrt{2}\sqrt{-1} = \sqrt{2}i$.
- Be careful! When replacing $\sqrt{-1}$ by i, the exponent rules for real numbers don't work. Example: $\sqrt{-2}\sqrt{-3} \neq \sqrt{(-2)(-3)}$
- To write square root of a negative number as an imaginary number: Separate the $\sqrt{-1}$ and then simplify the other square root.
- Complex numbers: They are of the form $a + bi$ where a and b are real numbers. a is called the real part and b is called the imaginary part of the number. If $a = 0$, then the complex number is an **imaginary number**. If $b = 0$, then the complex number is a real number.
- Why complex numbers? To be able to find roots of all polynomials. Examples: $x^2 + 1 = 0$, $x^2 + 4x + 5 = 0$ and ...
- Complex Plane is a two dimensional plane where each point associates with a complex number. The x-value of a point represent the real part of number and the y-value of the point is the imaginary part of the number. The number (a, b) represents complex number $a + bi$.

  Note that any point on x-axis is a real number and any point on y-axis is an imaginary number.



Operations on Complex Numbers

Let $a_1 + b_1 i$ and $a_2 + b_2 i$ be two complex numbers. Then

- **the sum of the two** is $\left(a_1 + b_1 i\right) + \left(a_2 + b_2 i\right) = (a_1 + a_2) + (b_1 + b_2)i$
- **the difference of the two** is $\left(a_1 + b_1 i\right) - \left(a_2 + b_2 i\right) = (a_1 - a_2) + (b_1 - b_2)i$
- **the product of the two** is

  $$\left(a_1 + b_1 i\right).\left(a_2 + b_2 i\right) = \underbrace{a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 \overset{-b_1 b_2}{i^2}}_{\text{Foil the two}} \underset{\text{Regroup}}{=} (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2)i$$

- **The complex conjugate** of $a + bi$ is $a - bi$. We use the complex conjugates to divide two complex numbers. Main property of complex conjugates is

  $$(a + bi) \cdot (a - bi) = a^2 + b^2$$

- **Dividing two complex numbers**: If $a_2 \neq 0$ or $b_2 \neq 0$, then $\dfrac{a_1 + b_1 i}{a_2 + b_2 i} = \left(\dfrac{a_1 + b_1 i}{a_2 + b_2 i}\right) . \left(\dfrac{a_2 - b_2 i}{a_2 - b_2 i}\right) =$ $\dfrac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)} = \dfrac{(a_1 a_2 + b_1 b_2) + (-a_1 b_2 + b_1 a_2)i}{a^2 + b^2} = \dfrac{(a_1 a_2 + b_1 b_2)}{a^2 + b^2} + \dfrac{(-a_1 b_2 + b_1 a_2)}{a^2 + b^2} i$

  Note that we used the complex conjugate to rewrite a fraction (division) in the standard form, $a + bi$, of a complex number.

- Power to integer exponent: For now, we only discuss the integer exponents and we discuss them in terms of multiplication. If $n > 0$ is a whole number then $(a + bi)^n = \underbrace{(a + bi)(a + bi) \cdots (a + bi)}_{n}$.

  When calculating this product, simplify using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$, $i^9 = i$ and ....

  Notice that higher powers of i reduce to power 0 or 1.

- How do we verify a complex/real number is a root for a polynomial/equation? Plug in that complex/real number in the polynomial/equation. If the polynomial is zero or equation is satisfied, then the number is a root.

1. Perform the following operations and express the result as a simplified (in standard form $a+bi$) complex number.

(a) $9+(3+11i)$

(b) $11i+(9+13i)$

(c) $(9+11i)+(13-3i)$

(d) $(13+3i)-(1-i)$

(e) $9\cdot(9+13i)$

(f) $(3i)\cdot(9+11i)$

(g) $(9+11i)\cdot(13+3i)$

(h) $\dfrac{9+11i}{3i}$

(i) $\dfrac{9+11i}{13}$

(j) $\dfrac{9+11i}{3+4i}$

2. Perform each of the following operations.

(a) i^{24}

(b) i^{41}

(c) i^{23}

(d) i^{26}

(e) i^{100}

(f) i^{101}

(g) i^{102}

(h) i^{103}

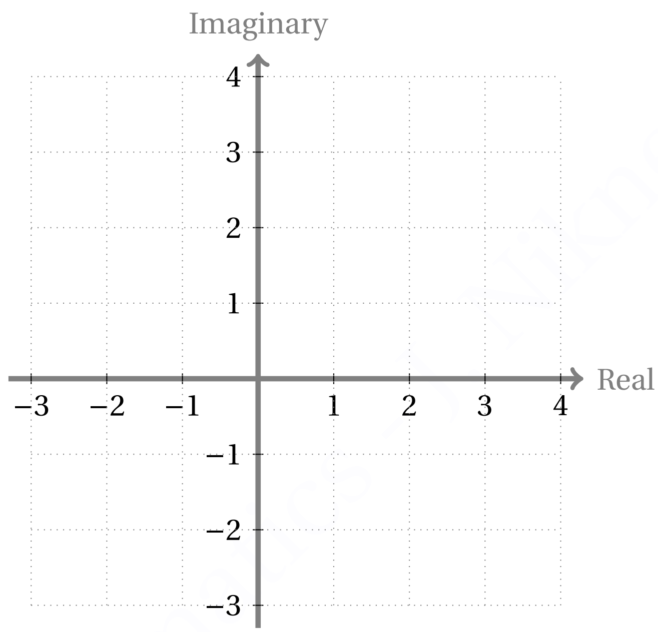
3. Evaluate the following algebraic expressions.

(a) If $f(x)=x^2-2x+2$, evaluate $f(2i)$.

(b) If $f(x)=x^2-2x+5$, evaluate $f(i+1)$.

4. Verify that $-i-1$ is a solution to $x^2+2x+2=0$.

5. Plot $-2+3i$ and $3-2i$ on the complex plane. Clearly mark each point.



6. Write $\dfrac{2-3i}{7+i}$ in the form $a+bi$.

(a) $11-23i$

(b) $\dfrac{11}{8}-\dfrac{23}{8}i$

(c) $-\dfrac{11}{50}-\dfrac{23}{50}i$

(d) $\dfrac{11}{50}-\dfrac{23}{50}i$

7. Solve $(x+11)^3 = 27$ for x, in complex numbers domain.

8. Solve $(x-11)^3 + 125 = 0$ for x, in complex numbers domain.

9. Solve $3x^4 - 13x^2 - 10 = 0$ for x, in the complex numbers domain.

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